Holographic charged fluid dual to third order Lovelock gravity

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We study the dual fluid on a finite cutoff surface outside the black brane horizon in the third order Lovelock gravity. Using nonrelativistic long-wavelength expansion, we obtain the incompressible Navier-Stokes equations of dual fluid with external force density on the finite cutoff surface. The viscosity to entropy density ratio η/s is independent of cutoff surface and does not get modification from the third order Lovelock gravity influence. The obtained ratio agrees with the results obtained by using other methods, such as the Kubo formula at the AdS boundary and the membrane paradigm at the horizon in the third order Lovelock gravity. These results can be related by Wilson renormalization group flow. However the kinematic viscosity receives correction from the third order Lovelock term. We show that the equivalence between the isentropic flow of the fluid and the radial component of the gravitational equation observed in the Einstein and Gauss-Bonnet gravities also holds in the third order Lovelock gravity. This generalization brings more understandings of relating the gravity theory to the dual fluid.

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I. INTRODUCTION

It was found nearly three decades ago that the excitations of a black hole horizon dissipate like those of a fluid with viscosity $\eta = 1/16\pi G$ [1–4]. Dividing η with the Bekenstein-Hawking entropy density $s = 1/4G$, it yields the dimensionless ratio $\eta/s = 1/4\pi$. Recently in studying the AdS/CFT correspondence, the ratio of the shear viscosity to entropy density of some field theories dual to the AdS Einstein gravity was again shown to be the universal value $1/4\pi$ [5]. Instead of the black hole horizon, in the study of AdS/CFT correspondence the ratio of η/s was obtained on the AdS boundary. Moreover, later then it was found that there exists a so-called fluid/gravity duality, which can be considered as a special limit of AdS/CFT, claiming that the dual boundary

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field theories can be described by hydrodynamics in the long-wavelength limit [6]. Considering that the Wilson renormalization group flow theory does not require an ultraviolet completion of quantum field theory, the authors in [7] introduced a finite cutoff r_c outside the horizon in a general class of black hole geometries and did not restrict in the asymptotically AdS background. Then, a precise mathematical relation between the incompressible Navier-Stokes equations in $(p + 1)$ dimensions and vacuum Einstein equations has been given in $(p + 2)$-dimensional Rindler bulk spacetime on an arbitrary cutoff surface $r = r_c$ outside the horizon [8, 9]. It is interesting that the shear viscosity over the entropy density of the fluid is still $1/4\pi$, independent of the cutoff surface. This study was extended to the AdS black brane [10–13]. Imposing the Petrov-like condition on the $\Sigma_c(r = r_c)$ in the near horizon limit, the incompressible Navier-Stokes equations (or modified equations) for a fluid living on the flat (or spatially curved) spacetime with one fewer dimensions have been demonstrated in [14–17]. The physics on a finite cutoff surface Σ_c with finite energy scale is appealing since it could be reached by experiments. The study of holography on the finite surface Σ_c may be helpful to understand the microscopic origin of gravity. Other recent works on the fluid/gravity correspondence can be found in [18–28].

The study of the fluid/gravity correspondence mainly concentrated on the Einstein-Hilbert action in the gravity side by relating the radial component of Einstein equations to the isentropy equation of the dual field theory. In the low-energy limit, string theories give rise to effective gravity models in higher dimensions which involve the higher powers of curvature terms. In Gauss-Bonnet gravity, the incompressible Navier-Stokes equations (with external force density) were obtained on a general cutoff surface Σ_c for the Rindler [29] and (charged) black brane metrics [10, 11]. Although the thermodynamic relation still holds, the universal relation of the ratio of shear viscosity to entropy density is broken and the diffusivity is changed due to the appearance of the Gauss-Bonnet factor. In this paper we will extend the study to a more general gravity theory with higher order curvature correction, the third order Lovelock gravity. We will follow the procedures in [10, 11, 29] and consider the external force coming from the Maxwell field in the system. We will introduce a finite cutoff surface outside the black hole horizon and discuss the forced fluids at the cutoff surface in the third order Lovelock gravity. We will obtain the incompressible Navier-Stokes equations of the dual fluid with external force density on Σ_c in equivalence with the radial component of gravitational field equations. We will examine whether the higher curvature correction in the third order Lovelock gravity influences the thermodynamic parameters, such as the ratio η/s, the kinematic viscosity, the Reynolds number and the diffusivity of the dual fluid. This will give us further understanding of relating the gravity theory to the dual fluid.

The outline of this paper is as follows. In Sec. II, we present the background with electromagnetic field in the third order Lovelock gravity and introduce the nonrelativistic long-wavelength expansion, where the bulk gravitational and Maxwell equations are solved to the second order of the expansion parameter. In Sec. III, we calculate the stress-energy tensor of the dual fluid through the Brown-York tensor on the cutoff surface and present the incompressible Navier-Stokes equation of the dual fluid in the third order Lovelock Maxwell case. In Sec. IV, we examine the ratio of shear viscosity to entropy density, the kinematic viscosity, the Reynolds number and the diffusivity of the dual fluid on the finite cutoff surface with the presence of higher order curvature corrections. We finally summarize our results in Sec. V.

II. NONRELATIVISTIC HYDRODYNAMIC EXPANSION

We consider the third order Lovelock gravity with electromagnetic field described by the action

$$\mathcal{I}_G = \frac{1}{16\pi G} \int d^n x \sqrt{-g} \left(-2\Lambda + R + \alpha_2 \mathcal{L}_2 + \alpha_3 \mathcal{L}_3 - 4\pi G F_{\mu\nu} F^{\mu\nu} \right), \tag{1}$$

where the negative cosmological constant $\Lambda = -(n-1)(n-2)/2l^2$ and l is the AdS radius. α_2 and α_3 are coefficients of Gauss-Bonnet term $\mathcal{L}_2$ and third order Lovelock term $\mathcal{L}_3$ respectively. The expressions of $\mathcal{L}_2$ and $\mathcal{L}_3$ are

$$\mathcal{L}_2 = R_{\mu\nu\sigma\kappa} R^{\mu\nu\sigma\kappa} - 4R_{\mu\nu} R^{\mu\nu} + R^2,$$

$$\mathcal{L}_3 = 2R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\rho\tau} R^{\rho\tau}{}_{\mu\nu} + 8R^{\mu\nu}{}_{\sigma\rho} R^{\sigma\kappa}{}_{\nu\tau} R^{\rho\tau}{}_{\mu\kappa} + 24 R^{\mu\nu\sigma\kappa} R_{\sigma\kappa\nu\rho} R^{\rho}{}_{\mu}$$

$$+ 3RR^{\mu\nu\sigma\kappa} R_{\mu\nu\sigma\kappa} + 24 R^{\mu\nu\sigma\kappa} R_{\sigma\mu} R_{\kappa\nu} + 16 R^{\mu\nu} R_{\nu\sigma} R^{\sigma}{}_{\mu} - 12 RR^{\mu\nu} R_{\mu\nu} + R^3.$$

Varying the action with respect to the metric tensor $g_{\mu\nu}$ and electromagnetic field A_μ, the equations of motion (EOM) for gravitational and electromagnetic fields are presented

$$W_{\mu\nu} = -\frac{(n-1)(n-2)}{2l^2} g_{\mu\nu} + G^{(1)}_{\mu\nu} + \alpha_2 G^{(2)}_{\mu\nu} + \alpha_3 G^{(3)}_{\mu\nu} + 8\pi G T_{\mu\nu} = 0, \tag{2}$$

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0, \tag{3}$$

where $T_{\mu\nu}$ is the electromagnetic stress tensor $\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - F_{\mu\alpha} F_\nu{}^\alpha$. In addition, $G^{(1)}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ is the Einstein tensor, and $G^{(2)}_{\mu\nu}$ and $G^{(3)}_{\mu\nu}$ are the Gauss-Bonnet and third order Lovelock tensors respectively

$$G^{(2)}_{\mu\nu} = 2 \left(R_{\mu\sigma\kappa\tau} R_\nu{}^{\sigma\kappa\tau} - 2R_{\mu\rho\nu\sigma} R^{\rho\sigma} - 2R_{\mu\sigma} R^\sigma{}_\nu + RR_{\mu\nu} \right) - \frac{1}{2} \mathcal{L}_2 g_{\mu\nu},$$

$$
\begin{aligned}
G_{\mu\nu}^{(3)} = {} & 3R_{\mu\nu}R^2 - 12RR_\mu{}^\sigma R_{\sigma\nu} - 12R_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} + 24R_\mu{}^\alpha R_\alpha{}^\beta R_{\beta\nu} - 24R_\mu{}^\alpha R^{\beta\sigma}R_{\alpha\beta\sigma\nu} \\
& + 3R_{\mu\nu}R_{\alpha\beta\sigma\kappa}R^{\alpha\beta\sigma\kappa} - 12R_{\mu\alpha}R_{\nu\beta\sigma\kappa}R^{\alpha\beta\sigma\kappa} - 12RR_{\mu\sigma\nu\kappa}R^{\sigma\kappa} + 6RR_{\mu\alpha\beta\sigma}R_\nu{}^{\alpha\beta\sigma} \\
& + 24R_{\mu\alpha\nu\beta}R_\sigma{}^\alpha R^{\sigma\beta} + 24R_{\mu\alpha\beta\sigma}R_\nu{}^\beta R^{\alpha\sigma} + 24R_{\mu\alpha\nu\beta}R_{\sigma\kappa}R^{\alpha\sigma\beta\kappa} - 12R_{\mu\alpha\beta\sigma}R^{\kappa\alpha\beta\sigma}R_{\kappa\nu} \\
& - 12R_{\mu\alpha\beta\sigma}R^{\alpha\kappa}R_{\nu\kappa}{}^{\beta\sigma} + 24R_\mu{}^{\alpha\beta\sigma}R_\beta{}^\kappa R_{\sigma\kappa\nu\alpha} - 12R_{\mu\alpha\nu\beta}R^\alpha{}_{\sigma\kappa\rho}R^{\beta\sigma\kappa\rho} \\
& - 6R_\mu{}^{\alpha\beta\sigma}R_{\beta\sigma}{}^{\kappa\rho}R_{\kappa\rho\alpha\nu} - 24R_{\mu\alpha}{}^{\beta\sigma}R_{\beta\rho\nu\lambda}R_\sigma{}^{\lambda\alpha\rho} - \frac{1}{2}\mathcal{L}_3 g_{\mu\nu}.
\end{aligned}
$$

For the third order Lovelock gravity, the nontrivial third order term $\mathcal{L}_3$ requires the dimension (n) of spacetime satisfying $n \geq 7$ [30, 31]. We take the unit AdS radius $l = 1$ in what follows for convenience.

Taking the ansatz of the charged black brane solution in the third order Lovelock gravity as

$$
ds_n^2 = -f(r)d\tau^2 + 2d\tau dr + r^2 dx_i dx^i, \tag{4}
$$

we will have [30, 31]

$$
f(r) = \frac{\tilde{\alpha}_2 r^2}{\tilde{\alpha}_3}\left[1 + \sqrt[3]{\sqrt{\gamma + \chi^2(r)} + \chi(r)} - \sqrt[3]{\sqrt{\gamma + \chi^2(r)} - \chi(r)}\right], \tag{5}
$$

$$
A = \sqrt{\frac{n-2}{8\pi(n-3)G}}\frac{q}{r^{n-3}}d\tau, \tag{6}
$$

by solving the EOM, where

$$
\gamma = \left(\frac{\tilde{\alpha}_3}{\tilde{\alpha}_2^2} - 1\right)^3, \quad \chi(r) = 1 - \frac{3\tilde{\alpha}_3}{2\tilde{\alpha}_2^2} - \frac{3\tilde{\alpha}_3^2}{2\tilde{\alpha}_2^3}\left(-1 + \frac{m}{r^{n-1}} - \frac{q^2}{r^{2n-4}}\right).
$$

We have introduced new parameters $\alpha_2 = \frac{\tilde{\alpha}_2}{(n-3)(n-4)}$ and $\alpha_3 = \frac{\tilde{\alpha}_3}{72\binom{n-3}{4}}$. For the convenience of the following discussion, considering that usually the higher order curvature corrections are small, we can expand the black brane solution Eq. (5) to the first order of $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ as

$$
f(r) = \Psi(r) + \frac{\tilde{\alpha}_2}{r^2}\Psi^2(r) - \frac{\tilde{\alpha}_3}{3r^4}\Psi^3(r) \tag{7}
$$

with $\Psi(r) = r^2 - \frac{m}{r^{n-3}} + \frac{q^2}{r^{2n-6}}$. The first two terms are exactly the expansion of the metric coefficient for the Gauss-Bonnet black brane solution to the first order of $\tilde{\alpha}_2$.

The integral constants m and q here are related to the gravitational mass $M = \frac{(n-2)mV_{n-2}}{16\pi G}$ and the total charge $Q^2 = \frac{2(n-2)(n-3)\pi q^2}{G}$ respectively. Moreover m in terms of the real root of $f(r_h) = 0$ is $m = r_h^{n-1} + \frac{q^2}{r_h^{n-3}}$. Then, the Hawking temperature T_h of the black brane is obtained

$$
\begin{aligned}
T_h &= \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi}\left[\frac{(n-1)m}{r_h^{n-2}} - \frac{2(n-2)q^2}{r_h^{2n-5}}\right] \\
&= \frac{1}{4\pi}\left[(n-1)r_h - \frac{(n-3)q^2}{r_h^{2n-5}}\right].
\end{aligned} \tag{8}
$$

Notice that these expressions for mass m and temperature T_h are independent of the Gauss-Bonnet and the third order Lovelock terms. For $T_h > 0$, we must satisfy the condition $r_h^{2n-4} > \frac{n-3}{n-1}q^2$. For the extreme black brane case with $T_h \to 0$, $r_h^{2n-4} = \frac{n-3}{n-1}q^2$.

The induced metric on the cutoff surface $\Sigma_c(r = r_c)$ outside the horizon r_h with the intrinsic coordinates $\tilde{x}^a \sim \left(\tilde{\tau} = \sqrt{f(r_c)}\tau, \tilde{x}^i = r_c x^i \right)$ is

$$
\begin{aligned}
ds_{n-1}^2 &= \gamma_{ab}dx^a dx^b = -f(r_c)d\tau^2 + r_c^2 dx_i dx^i \\
&= -d\tilde{\tau}^2 + \delta_{ij}d\tilde{x}^i d\tilde{x}^j.
\end{aligned}
\tag{9}
$$

We require the metric Eq. (9) flat when perturbing the bulk metric Eq. (4) and will investigate the dual fluid living on the $\Sigma_c(r = r_c)$.

Let us begin by describing the setup. Following [9], we consider the diffeomorphism transformations satisfying the three conditions: (i) the induced metric on the Σ_c is fixed; (ii) the stress-energy tensor on the Σ_c takes the form of that of perfect fluid; (iii) metrics after diffeomorphism transformations remain stationary and homogeneous in the (τ, x^i) coordinates. Then there are two allowed finite diffeomorphism transformations. The first diffeomorphism transformation is a Lorentz boost with constant boost parameter β_i

$$
\begin{cases}
\sqrt{f(r_c)}\tau \to \gamma(\sqrt{f(r_c)}\tau - \beta_i r_c x^i), \\
r_c x^i \to r_c x^i + (\gamma - 1)\frac{\beta^i \beta_j}{\beta^2}x^j - \gamma\beta^i\sqrt{f(r_c)}\tau,
\end{cases}
\tag{10}
$$

where $u^a = \frac{(1, \beta^i)}{\sqrt{f(r_c) - v^2}}$, $\gamma = \left(1 - \beta^2\right)^{-1/2}$ and $\beta^i = \frac{r_c}{\sqrt{f(r_c)}}v^i$. The second is a transformation of r and associated rescalings of τ and x^a

$$
r \to (1 - P)r, \quad \tau \to \sqrt{\frac{f(r_c)}{f\left[(1 - P)r_c\right]}}\tau, \quad x^i \to \frac{r_c}{(1 - P)r_c}x^i.
\tag{11}
$$

After these transformations, we can obtain the transformed bulk metric and electromagnetic field and they remain the exact solution of EOM.

After promoting v_i and P to be functions of boundary coordinates, $v_i = v_i(\tau, x^i)$ and $P = P(\tau, x^i)$, the transformed bulk metric and electromagnetic field are no longer the exact solution of EOM. Taking the so-called nonrelativistic long-wavelength expansion parametrized by $\epsilon \to 0$

$$
\partial_\tau \sim \epsilon^2, \quad \partial_i \sim \epsilon, \quad \partial_r \sim \epsilon^0
\tag{12}
$$

together with scaling $P \sim \epsilon^2$ and $v_i \sim \epsilon$, the transformed bulk metric up to ϵ^2 is [10, 11]

$$
\begin{aligned}
ds^2 = & -f(r)d\tau^2 + 2drd\tau + r^2 dx_i dx^i \\
& -2r^2 \left(1 - \frac{r_c^2 f(r)}{r^2 f(r_c)}\right) v_i dx^i d\tau - \frac{2r_c^2 v_i}{f(r_c)} dx^i dr \\
& +r^2 \left(1 - \frac{r_c^2 f(r)}{r^2 f(r_c)}\right) \left(v^2 d\tau^2 + \frac{r_c^2 v_i v_j}{f(r_c)} dx^i dx^j\right) + \frac{r_c^2 v^2}{f(r_c)} drd\tau \\
& +f(r) \left(\frac{rf'(r)}{f(r)} - \frac{r_c f'(r_c)}{f(r_c)}\right) Pd\tau^2 + \left(\frac{r_c f'(r_c)}{f(r_c)} - 2\right) Pdrd\tau \\
& +2r^2 F(r)\partial_{(i} v_{j)} dx^i dx^j + \mathcal{O}(\epsilon^3),
\end{aligned}
\tag{13}
$$

where the terms of last three lines are all of order ϵ^2. Meanwhile, the correction term $2r^2 F(r)\partial_{(i} v_{j)} dx^i dx^j$ is added to cancel the source terms at order ϵ^2 due to the spatial $SO(n-2)$ rotation symmetry of black brane background [6] and the gauge $F(r_c) = 0$ is chosen to keep the induced metric γ_{ab} invariant. Henceforth the above total metric can solve the gravity equations up to order ϵ^2 and $F(r)$ will be determined below using the gravity equations. In addition, the transformed electromagnetic field turns to

$$
\begin{aligned}
A_\mu dx^\mu = & \sqrt{\frac{2n-4}{n-3}} \frac{q}{r^{n-3}} \left[d\tau - \frac{r_c^2}{f(r_c)} v_i dx^i + \frac{r_c^2}{2f(r_c)} v^2 d\tau \right. \\
& \left. + (n-3)Pd\tau + \frac{f'(r_c)}{2f(r_c)} r_c Pd\tau\right] + \mathcal{O}(\epsilon^3)
\end{aligned}
\tag{14}
$$

and the nonvanishing components of the electromagnetic field up to ϵ^2 are

$$
\begin{aligned}
F_{\tau r} = & \sqrt{\frac{(n-2)(n-3)}{8\pi G}} \frac{q}{r^{n-2}} \left[1 + \frac{r_c^2 v^2}{2f(r_c)} + \left((n-3) + \frac{f'(r_c)r_c}{2f(r_c)}\right) P\right], \\
F_{ir} = & -\sqrt{\frac{(n-2)(n-3)}{8\pi G}} \frac{r_c^2 v_i}{f(r_c)}, \quad (i = 3, ...n).
\end{aligned}
\tag{15}
$$

In [7], the electromagnetic degrees of freedom have been considered and then there exists a boundary current dual to the bulk electromagnetic field.

It is interesting to note that the perturbed electromagnetic field Eq. (15) automatically satisfies the Maxwell equation Eq. (3) up to ϵ^2, provided that one additionally imposes a constraint condition $\partial_i v^i = 0$ which is equivalent to $\tilde{\partial}_i \beta^i = 0$. As we will see later, this condition turns out to be just the constraint equation at order ϵ^2. So it needs no correction terms to be added to electromagnetic field Eq. (15) at order ϵ^2. As to the Lovelock gravity equation Eq. (2), with this constraint condition $\tilde{\partial}_i \beta^i = 0$, $F(r)$ can be solved as

$$
F(r) = \int_r^{r_c} \left[\left(1 - \frac{C}{\vartheta r^{n-2}}\right) \frac{1}{f(r)}\right] dr,
\tag{16}
$$

where

$$\vartheta = 1 - 2\tilde{\alpha}_2 \left(\frac{f(r)}{r^2} + \frac{r}{n-3} \left(\frac{f(r)}{r^2} \right)' \right) + \frac{\tilde{\alpha}_3 f(r)}{r^2} \left(\frac{f(r)}{r^2} + \frac{r}{n-5} \left(\frac{f(r)}{r^2} \right)' \right).$$

The integration constant C is determined as $C = r_h^{n-2} - \frac{2\tilde{\alpha}_2}{n-3} r_h^{n-3} f'(r_h)$ in order to keep the function $F(r)$ regular at the horizon r_h [the largest root of $f(r_h) = 0$]. With $f'(r_h) = (n-1)r_h - \frac{(n-3)q^2}{r_h^{2n-5}}$, we can rewrite Eq. (16) to a useful form for the following discussions

$$\left(1 + F'(r)f(r) \right) \vartheta r^{n-2} = r_h^{n-2} \left[1 - 2\tilde{\alpha}_2 \left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}} \right) \right]. \tag{17}$$

Obviously the function $\left(1 + F'(r)f(r) \right) \vartheta r^{n-2}$ cannot be influenced by the cutoff surface and coefficient $\tilde{\alpha}_3$ of the third order Lovelock term.

III.   INCOMPRESSIBLE CHARGED FLUID ON THE CUTOFF SURFACE Σ_c

Now we study the fluid living on the surface Σ_c dual to the bulk configuration Eqs. (13)(14). According to fluid/gravity duality, the Brown-York tensor on the Σ_c can be identified as the stress-energy tensor of the dual fluid.

For a spacetime with boundary in Einstein gravity, the equations of motion are derived from the Einstein-Hilbert action with a Gibbons-Hawking boundary term [32]. In the third order Lovelock gravity, some appropriate surface terms [33] are also needed to be included to the action Eq. (1)

$$\mathcal{I} = \mathcal{I}_G + \frac{1}{8\pi G} \int_{\Sigma_c} d^{n-1}x \sqrt{-\gamma} \left(K + 2\alpha_2 J + 3\alpha_3 H + \mathcal{C} \right), \tag{18}$$

where $\gamma_{ab} = g_{ab} - n_a n_b$ is an induced metric on Σ_c, K is the trace of extrinsic curvature tensor K_{ab} on Σ_c and is defined by $K_{ab} = \gamma^c{}_a \nabla_c n_b$. Furthermore, J and H are the traces of

$$J_{ab} = \frac{1}{3} \left(2K K_{ac} K_b^c + K_{cd} K^{cd} K_{ab} - 2K_{ac} K^{cd} K_{db} - K^2 K_{ab} \right),$$

$$\begin{aligned} H_{ab} &= \frac{1}{5} \left[\left(K^4 - 6K^2 K^{cd} K_{cd} + 8K K_{cd} K^d{}_e K^{ec} - 6K_{cd} K^{de} K_{ef} K^{fc} + 3 \left(K_{cd} K^{cd} \right)^2 \right) K_{ab} \right. \\ &\quad - \left(4K^3 - 12K K_{ed} K^{ed} + 8K_{de} K^e{}_f K^{fd} \right) K_{ac} K^c{}_b - 24K K_{ac} K^{cd} K_{de} K^e{}_b \\ &\quad + \left. \left(12K^2 - 12K_{ef} K^{ef} \right) K_{ac} K^{cd} K_{db} + 24 K_{ac} K^{cd} K_{de} K^{ef} K_{bf} \right]. \end{aligned} \tag{19}$$

Here $\mathcal{C}$ is a constant which can be fixed when the cutoff surface goes to the AdS boundary. Using the Brown-York method [34], the Brown-York tensor T_{ab}^{BY} on the Σ_c can be derived [35–37]

$$\begin{aligned} T_{ab}^{BY} &= \frac{1}{8\pi G} \left[K\gamma_{ab} - K_{ab} - 2\alpha_2 \left(3J_{ab} - J\gamma_{ab} \right) \right. \\ &\quad \left. - 3\alpha_3 \left(5H_{ab} - H\gamma_{ab} \right) + \mathcal{C}\gamma_{ab} \right], \end{aligned} \tag{20}$$

Plugging the perturbed metric Eq. (13) into Eq. (20), the Brown-York tensor T_{ab}^{BY} of the dual fluid with $\alpha_2 = \frac{\tilde{\alpha}_2}{(n-3)(n-4)}$ and $\alpha_3 = \frac{\tilde{\alpha}_3}{72\binom{n-3}{4}}$ in the $\tilde{x}^a \sim (\tilde{\tau}, \tilde{x}^i)$ coordinates can be described as

$$\tilde{T}_{ab}^{BY} = \tilde{T}_{ab}^{(0)} + \tilde{T}_{ab}^{(1)} + \tilde{T}_{ab}^{(2)} + \mathcal{O}(\epsilon^3), \tag{21}$$

where

$$
\begin{aligned}
8\pi G \tilde{T}_{ab}^{(0)} d\tilde{x}^a d\tilde{x}^b &= \left[\frac{(n-2)f(r_c)^{1/2}}{r_c} \left(-\frac{\tilde{\alpha}_3 f^2(r_c)}{5r_c^4} + \frac{2\tilde{\alpha}_2 f(r_c)}{3r_c^2} - 1 \right) - \mathcal{C} \right] d\tilde{\tau}^2 \\
&\quad + \left[\frac{1}{\sqrt{f(r_c)}} \left(\frac{f'(r_c)}{2} + \frac{(n-3)f(r_c)}{r_c} - \tilde{\alpha}_2 f(r_c) \left(\frac{f'(r_c)}{r_c^2} + \frac{2(n-5)f(r_c)}{3r_c^3} \right) \right. \right. \\
&\quad \left. \left. + \tilde{\alpha}_3 f^2(r_c) \left(\frac{f'(r_c)}{2r^4} + \frac{(n-7)f(r_c)}{5r^5} \right) \right) + \mathcal{C} \right] d\tilde{x}_i d\tilde{x}^i,
\end{aligned}
\tag{22}
$$

$$8\pi G \tilde{T}_{ab}^{(1)} d\tilde{x}^a d\tilde{x}^b = \left(-\frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4} + \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} - 1 \right) \left(\frac{f(r)}{r^2} \right)'_c \frac{r_c^2 \beta_i}{\sqrt{f(r_c)}} d\tilde{x}^i d\tilde{\tau}, \tag{23}$$

$$
\begin{aligned}
8\pi G \tilde{T}_{ab}^{(2)} d\tilde{x}^a d\tilde{x}^b &= \left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4} \right) \left(\frac{f(r)}{r^2} \right)'_c \frac{r_c^2}{2\sqrt{f(r_c)}} \left[\left((n-2)P + \beta^2 \right) d\tilde{\tau}^2 \right. \\
&\quad \left. + \beta_i \beta_j d\tilde{x}^i d\tilde{x}^j + \kappa P d\tilde{x}_i d\tilde{x}^i \right] - \frac{\vartheta_c}{2} \left[1 + f(r_c) F'(r) \right] \tilde{\partial}_{(i} \beta_{j)} d\tilde{x}^i d\tilde{x}^j
\end{aligned}
\tag{24}
$$

with

$$
\begin{aligned}
\kappa &= \frac{r_c^3 \left(r_c^4 + 2\tilde{\alpha}_2 r_c^2 f(r_c) - \tilde{\alpha}_3 f^2(r_c) \right)}{2f(r_c) \left(r_c^4 - 2\tilde{\alpha}_2 r_c^2 f(r_c) + \tilde{\alpha}_3 f^2(r_c) \right)} \left(\frac{f(r)}{r^2} \right)'_c \\
&\quad - (n-1) - r_c \left(\frac{f(r)}{r^2} \right)''_c \Big/ \left(\frac{f(r)}{r^2} \right)'_c, \\
\vartheta_c &= 1 - 2\tilde{\alpha}_2 \left(\frac{f(r_c)}{r_c^2} + \frac{r_c}{n-3} \left(\frac{f(r)}{r^2} \right)'_c \right) + \frac{\tilde{\alpha}_3 f(r_c)}{r_c^2} \left(\frac{f(r_c)}{r_c^2} + \frac{r_c}{n-5} \left(\frac{f(r)}{r^2} \right)'_c \right).
\end{aligned}
$$

On the fluid side, we know that the general form of the stress-energy tensor of a (relativistic) viscous fluid at the first order gradient expansion in Minkowski background is

$$\tilde{T}_{ab} = (\tilde{p} + \tilde{\rho}) \tilde{u}_a \tilde{u}_b + \tilde{p}\tilde{g}_{ab} - 2\eta \tilde{\sigma}_{ab} - \zeta \tilde{\theta} (\tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b) \tag{25}$$

with $\tilde{u}^a = \gamma(1, \beta^i)$, $\tilde{\rho}$ the energy density, $\tilde{p}$ the pressure, $\tilde{\sigma}_{ab}$ the shear and $\tilde{\theta} = \tilde{\partial}_a \tilde{u}^a$ the expansion. In the above stress-energy tensor under the nonrelativistic long-wavelength expansion, we have $\tilde{\theta} = 0$ by the incompressibility condition $\tilde{\partial}_a \tilde{u}^a \sim \tilde{\partial}_i \beta^i = 0$ at ϵ^2, which renders the term $\zeta \tilde{\theta}(\tilde{g}_{ab} + \tilde{u}_a \tilde{u}_b)$ to vanish and $\tilde{\sigma}_{ab} = \frac{1}{2}(\tilde{\partial}_a \tilde{u}_b + \tilde{\partial}_b \tilde{u}_a)$. Then up to ϵ^2, we have

$$
\begin{aligned}
\tilde{T}_{\tau\tau} &= \tilde{\rho} + (\tilde{p} + \tilde{\rho}) \beta^2, \quad \tilde{T}_{\tau i} = -(\tilde{p} + \tilde{\rho}) \beta_i, \\
\tilde{T}_{ij} &= (\tilde{p} + \tilde{\rho}) \beta_i \beta_j + \tilde{p}\delta_{ij} - \eta(\tilde{\partial}_i \tilde{\beta}_j + \tilde{\partial}_j \tilde{\beta}_i).
\end{aligned}
\tag{26}
$$

Comparing with the expression of Brown-York tensor $\tilde{T}^{BY}$ Eq. (22), the energy density $\tilde{\rho}_0$ and pressure $\tilde{p}_0$ of the dual fluid at order ϵ^0 take the following form

$$\begin{cases} \tilde{\rho}_0 = \frac{(n-2)\sqrt{f(r_c)}}{8\pi G r_c}\left(-\frac{\tilde{\alpha}_3 f^2(r_c)}{5r_c^4} + \frac{2\tilde{\alpha}_2 f(r_c)}{3r_c^2} - 1\right) - \frac{\mathcal{C}}{8\pi G}, \\ \tilde{p}_0 = \frac{1}{8\pi G\sqrt{f(r_c)}}\left(\frac{f'(r_c)}{2} + \frac{(n-3)f(r_c)}{r_c} - \tilde{\alpha}_2 f(r_c)\left(\frac{f'(r_c)}{r_c^2} + \frac{2(n-5)f(r_c)}{3r_c^3}\right)\right. \\ \left. + \tilde{\alpha}_3 f^2(r_c)\left(\frac{f'(r_c)}{2r^4} + \frac{(n-7)f(r_c)}{5r^5}\right)\right) + \frac{\mathcal{C}}{8\pi G}. \end{cases} \tag{27}$$

Even though there exists the constant $\mathcal{C}$, the combination $\omega = \tilde{\rho}_0 + \tilde{p}_0$ denotes

$$\omega = \frac{r_c^2}{16\pi G\sqrt{f(r_c)}}\left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4}\right)\left(\frac{f(r)}{r^2}\right)'_c. \tag{28}$$

From Eqs. (24)(26), the energy density $\tilde{\rho}_c$ and the pressure $\tilde{p}_c$ to the order ϵ^2 are corrected to be

$$\begin{cases} \tilde{\rho}_c = \tilde{\rho}_0 + \frac{(n-2)r_c^2 P}{16\pi G\sqrt{f(r_c)}}\left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4}\right)\left(\frac{f(r)}{r^2}\right)'_c, \\ \tilde{p}_c = \tilde{p}_0 + \frac{r_c^2 \kappa P}{16\pi G\sqrt{f(r_c)}}\left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4}\right)\left(\frac{f(r)}{r^2}\right)'_c \end{cases} \tag{29}$$

and the shear viscosity η is given by

$$\eta = \frac{\vartheta_c}{16\pi G}\left[1 + f(r_c)F'(r_c)\right]. \tag{30}$$

In addition, we can also define P_r as pressure density with [38]

$$P_r = \frac{\tilde{p}_c - \tilde{p}_0}{\tilde{\rho}_0 + \tilde{p}_0} = \frac{\tilde{p}_c - \tilde{p}_0}{\omega} = \kappa P. \tag{31}$$

For the flat cutoff surface Σ_c, the Gauss-Codazzi equations take the following forms

$$\begin{cases} R_{\mu\nu\rho\sigma}\gamma^\mu_a\gamma^\nu_b\gamma^\rho_c\gamma^\sigma_d = K_{bc}K_{ad} - K_{ac}K_{bd}, \\ R_{a\nu\rho\sigma}n^a\gamma^\nu_b\gamma^\rho_c\gamma^\sigma_d = \partial_d K_{bc} - \partial_c K_{bd}, \\ R^a{}_{\nu c\mu}\gamma^c_a\gamma^\nu_b\gamma^\mu_d = K_{bc}K^c_d - KK_{bd}. \end{cases} \tag{32}$$

With the aid of these Gauss-Codazzi equations, the conservation equations of the Brown-York tensor on the Σ_c-the so-called momentum constraint-can be deduced from the equation of motion Eq. (2) of the third order Lovelock gravity

$$-2\left(G^{(1)}_{\mu\nu} + \alpha_2 G^{(2)}_{\mu\nu} + \alpha_3 G^{(3)}_{\mu\nu}\right)n^\mu\gamma^\nu_b = 16\pi G T_{\mu\nu}n^\mu\gamma^\nu_b,$$

$$\Longrightarrow \tilde{\partial}^a\tilde{T}^{BY}_{ab} = T_{\mu b}n^\mu, \tag{33}$$

where n^μ is the unit normal vector of Σ_c. Taking index $b = \tau$, the temporal component of the momentum constraint at ϵ^2 reads as

$$\tilde{\partial}^a\tilde{T}^{BY}_{a\tau} = T^\mu_\tau n_\mu = \frac{1}{\sqrt{f(r_c)}}T^r_\tau = 0,$$

$$\Rightarrow \left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4}\right)\left(\frac{f(r)}{r^2}\right)'_c \frac{r_c^2}{8\pi G\sqrt{f(r_c)}}\tilde{\partial}_i\beta^i = 0, \tag{34}$$

which implies the dual fluid is incompressible.

Taking index $b = j$, the spatial component of the momentum constraint up to ϵ^3 is given by

$$
\begin{aligned}
\tilde{\partial}^a \tilde{T}_{aj}^{BY} &= T_{\mu j} n^\mu = F_{ja} J^a, \\
\Rightarrow \quad & -\frac{r_c^3}{16\pi G \sqrt{f(r_c)}} \left\{ -\left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4}\right) \left(\frac{f(r)}{r^2}\right)_c' \left(\tilde{\partial}_\tau \beta_j + \beta^a \tilde{\partial}_a \beta_j + \kappa \tilde{\partial}_j P\right) \right. \\
& \left. + \frac{\sqrt{f(r_c)}}{r_c^2} \vartheta_c \left[1 + f(r_c) F'(r_c)\right] \tilde{\partial}^2 \beta_j \right\} = f_j,
\end{aligned}
\tag{35}
$$

where $f_j = F_{ja} J^a$ as the external force density. The dual charged current here defined as $J^a = n_\mu F^{a\mu}$ on the Σ_c gives $J^\tau = n_r F^{r\tau}$ and $J^i = n_r F^{ri}$, and then we gain the charged current conservation law $\partial_a J^a = 0$. This result is obvious that the conservation law of the boundary current J^a just coincides with the incompressibility condition $\tilde{\partial}_i \beta^i = 0$ for the constant dual charge density. Therefore, consider these expressions for ω Eq. (28) and η Eq. (30) and order that the so-called kinematic viscosity ν equals $\frac{\eta}{\omega}$; the spatial component of the momentum constraint Eq. (35) finally becomes the incompressible charged Navier-Stokes equations

$$
\tilde{\partial}_\tau \beta_j + \beta^b \tilde{\partial}_b \beta_j + \tilde{\partial}_j P_r - \nu \tilde{\partial}^2 \beta_j = \tilde{f}_j, \quad \tilde{\partial}_i \beta^i = 0,
\tag{36}
$$

where the external force density reads $\tilde{f}_j = \frac{f_j}{r_c \omega}$.

IV. PHYSICAL PROPERTIES OF THE HOLOGRAPHIC FLUID ON THE CUTOFF SURFACE Σ_c

Now let us discuss the physical properties of the incompressible charged Navier-Stokes fluid on the surface Σ_c. We can compute the trace of the stress-energy tensor $\tilde{T}_{ab}$, namely the Brown-York tensor $\tilde{T}_{ab}^{BY}$ on the Σ_c, in the $\tilde{x}^a \sim (\tilde{\tau}, \tilde{x}^i)$ coordinates up to ϵ^2

$$
\begin{aligned}
\tilde{T}_c &= \tilde{T}_{ab}^{BY} \tilde{\gamma}^{ab} = \frac{1}{8\pi G} \left[(n-2) K + \frac{2\tilde{\alpha}_2 J}{n-3} \right. \\
& \left. + \frac{\tilde{\alpha}_3 H}{(n-3)(n-4)(n-5)} + (n-1)\mathcal{C} \right],
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
K &= \frac{rf'(r_c) + 2(n-2)f(r_c)}{2r_c\sqrt{f(r_c)}} + \frac{P}{4r_c f^{3/2}(r_c)} \Big[4(n-2)f^2(r_c) \\
&\quad + r_c^2 f'^2(r_c) - 2r_c f(r_c)\left((n-2)f'(r_c) + r_c f''(r_c)\right) \Big], \\
J &= (n-2)(n-3) \Bigg[-\frac{\sqrt{f(r_c)}}{6r_c^3}\left(2(n-4)f(r_c) + 3r_c f'(r_c)\right) + \frac{P}{4r_c^3\sqrt{f(r_c)}}\left(4(4-n)f^2(r_c)\right) \\
&\quad + r_c^2 f'^2(r_c) + 2r_c f(r_c)\left((n-6)f'(r_c) + r_c f''(r_c)\right)\Big) \Bigg], \\
H &= (n-2)(n-3)(n-4)(n-5) \Bigg[\frac{f^{3/2}(r_c)}{10r_c^5}\left(2(n-6)f(r_c) + 5r_c f'(r_c)\right) \\
&\quad + \frac{\sqrt{f(r_c)}P}{4r_c^5}\left(4(n-6)f^2(r_c) - 3r_c^2 f'^2(r_c) - 2r_c f(r_c)\left((n-10)f'(r_c) + r_c f''(r_c)\right)\right) \Bigg]. \quad (38)
\end{aligned}
$$

In general, the trace $\tilde{T}_c$ of the stress-energy tensor $\tilde{T}_{ab}$ does not vanish. For the boundary at infinity $r_c \to \infty$, some surface counterterms are needed to remove the divergence in the stress-energy tensor. From the black brane solution Eq. (5), one can take $r \to \infty$

$$
\frac{f(r)}{r^2} \to \frac{\tilde{\alpha}_2}{\tilde{\alpha}_3}\left[1 + \sqrt[3]{\sqrt{\gamma + \chi^2(\infty)} + \chi(\infty)} - \sqrt[3]{\sqrt{\gamma + \chi^2(\infty)} - \chi(\infty)}\right] \quad (39)
$$

with $\gamma = \left(\frac{\tilde{\alpha}_3}{\tilde{\alpha}_2^2} - 1\right)^3$ and $\chi(\infty) = 1 - \frac{3\tilde{\alpha}_3}{2\tilde{\alpha}_2^2} + \frac{3\tilde{\alpha}_3^2}{2\tilde{\alpha}_2^3}$, and then define the effective AdS radius l_e as

$$
\frac{1}{l_e^2} = \frac{\tilde{\alpha}_2}{\tilde{\alpha}_3}\left[1 + \sqrt[3]{\sqrt{\gamma + \chi^2(\infty)} + \chi(\infty)} - \sqrt[3]{\sqrt{\gamma + \chi^2(\infty)} - \chi(\infty)}\right]. \quad (40)
$$

Since the stress-energy tensor $\tilde{T}_{ab}$ is traceless on the boundary at infinity, Eq. (38) takes the form when $r_c \to \infty$

$$
K = \frac{n-1}{l_e}, \quad J = -\frac{(n-1)(n-2)(n-3)}{3l_e^3},
$$
$$
H = \frac{(n-1)(n-2)(n-3)(n-4)(n-5)}{5l_e^5}. \quad (41)
$$

Therefore, the constant $\mathcal{C}$ is obtained

$$
\mathcal{C} = (n-2)\left(-\frac{1}{l_e} + \frac{2\tilde{\alpha}_2}{3l_e^3} - \frac{\tilde{\alpha}_3}{5l_e^5}\right). \quad (42)
$$

We can expand $\mathcal{C}$ to the first order of $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ so that

$$
\mathcal{C} = (n-2)\left(-1 + \frac{\tilde{\alpha}_2}{6} - \frac{\tilde{\alpha}_3}{30}\right). \quad (43)
$$

Notice that the constant $\mathcal{C}$ receives a negative correction from the third order Lovelock gravity. Consider $\tilde{\alpha}_3 \to 0$; then $\mathcal{C}$ equals $-3 + \alpha_2$ for $n = 5$ which is consistent with the expansion of

constant $\mathcal{C}$ to the first order of α_2 in Gauss-Bonnet gravity [10]. In the Einstein gravity, $\mathcal{C} = -3$ in five dimensions [10] which is consistent with our result by taking $\tilde{\alpha}_2$ and $\tilde{\alpha}_3 \to 0$.

In third order Lovelock gravity, considering the higher order curvature contribution, it is more appropriate to use the Wald entropy to describe the thermodynamical properties of the black hole. However, in our case since we are considering the black brane with planar horizon, the Wald entropy reduces to the Bekenstein-Hawking entropy $S = \frac{r_h^{n-2}}{4G}$ [30, 31]. Using the metric Eq. (9), we consider a quotient under shift of x^i, $x^i \sim x^i + n^i$ with $n^i \in Z$. Then, the spatial R^{n-2} on the Σ_c turns out to be an $(n-2)$-tours T^{n-2} with r_c-dependent volume $V_{n-2}(r_c) = r_c^{n-2}$. Therefore, the entropy density on the Σ_c is described by $S/V_{n-2}(r_c)$ in the form

$$s_c = \frac{1}{4G} \frac{r_h^{n-2}}{r_c^{n-2}}. \tag{44}$$

The local temperature T_c on the Σ_c is identified as the temperature of the dual fluid. With the Tolman relation and Eq. (8), we get the local temperature T_c

$$T_c = \frac{T_h}{\sqrt{f(r_c)}} = \frac{1}{4\pi\sqrt{f(r_c)}} \left[(n-1)\, r_h - \frac{(n-3)\, q^2}{r_h^{2n-5}} \right]. \tag{45}$$

The term $\frac{1}{\sqrt{f(r_c)}}$ can be expanded in the first order of $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ in the form

$$\frac{1}{\sqrt{f(r_c)}} = \Psi^{-1/2}(r_c) - \frac{\tilde{\alpha}_2}{2r_c^2}\Psi^{1/2}(r_c) + \frac{\tilde{\alpha}_3}{6r_c^4}\Psi^{3/2}(r_c) \tag{46}$$

where $\Psi(r_c) = r_c^2 - \frac{r_h^{n-1}}{r_c^{n-3}} + \frac{q^2}{r_c^{n-3}}\left(\frac{1}{r_c^{n-3}} - \frac{1}{r_h^{n-3}} \right)$, so that the local temperature T_c can be expressed as

$$T_c = \frac{1}{4\pi} \left[\Psi^{-1/2}(r_c) - \frac{\tilde{\alpha}_2}{2r_c^2}\Psi^{1/2}(r_c) + \frac{\tilde{\alpha}_3}{6r_c^4}\Psi^{3/2}(r_c) \right] \left[(n-1)\, r_h - \frac{(n-3)\, q^2}{r_h^{2n-5}} \right]. \tag{47}$$

It is clear that the Gauss-Bonnet and third order Lovelock gravity factors appear in the local temperature expression.

We define the chemical potential μ_c as $\mu_c = \frac{n-2}{8\pi G \sqrt{f(r_c)}}(\frac{q}{r_h^{n-3}} - \frac{q}{r_c^{n-3}})$ and the charge density $q_c = \frac{q}{V_{n-2}(r_c)}$ with $\frac{q}{r_c^{n-2}}$ on the Σ_c. Then the thermodynamic relation can be verified

$$\omega - s_c T_c = q_c \mu_c \tag{48}$$

with ω expressed in Eq. (28)

$$
\begin{aligned}
\omega &= \frac{r_c^2}{16\pi G \sqrt{f(r_c)}} \left(1 - \frac{2\tilde{\alpha}_2 f(r_c)}{r_c^2} + \frac{\tilde{\alpha}_3 f^2(r_c)}{r_c^4}\right) \left(\frac{f(r)}{r^2}\right)'_c \\
&= \frac{1}{8\pi G \sqrt{f(r_c)}} \left[\frac{(n-1)\, m}{2r_c^{n-2}} - \frac{(n-2)\, q^2}{r_c^{2n-5}} \right] \\
&= \frac{1}{8\pi G \sqrt{f(r_c)}} \left[\frac{(n-1)r_h^{n-1}}{2r_c^{n-2}} + \frac{(n-1)q^2}{2r_c^{n-2}r_h^{n-3}} - \frac{(n-2)q^2}{r_c^{2n-5}} \right]. \tag{49}
\end{aligned}
$$

The shear viscosity $\eta = \frac{\vartheta_c}{16\pi G}\left[1 + f(r_c)F'(r_c)\right]$ can be written as

$$16\pi G \eta r_c^{n-2} = r_h^{n-2}\left[1 - 2\tilde{\alpha}_2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]$$

$$\Rightarrow \eta = \frac{1}{16\pi G}\frac{r_h^{n-2}}{r_c^{n-2}}\left[1 - 2\tilde{\alpha}_2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right], \tag{50}$$

where Eq. (17) has been employed. When $\frac{n-1}{n-3} = \frac{q^2}{r_h^{2n-4}}$, we see that the higher curvature corrections do not influence the shear viscosity and we always have $\eta = \frac{1}{16\pi G}\frac{r_h^{n-2}}{r_c^{n-2}}$ [10, 11]. For the case $\frac{n-1}{n-3} \neq \frac{q^2}{r_h^{2n-4}}$, we have the positive shear viscosity provided that $\tilde{\alpha}_2 < \left[2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]^{-1}$. When $\tilde{\alpha}_2 = \left[2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]^{-1}$, we have the perfect fluid with vanishing shear viscosity. Here we focus on the case with $\eta > 0$. The ratio of the shear viscosity to entropy density is obtained as

$$\frac{\eta}{s_c} = \frac{1}{4\pi}\left[1 - 2\tilde{\alpha}_2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]. \tag{51}$$

Obviously the ratio $\frac{\eta}{s_c}$ is independent of the surface cutoff. This is consistent with the results in Einstein and Gauss-Bonnet gravity [10, 11]. Our ratio $\frac{\eta}{s_c}$ does not change under Wilson renormalization group (RG) flow and is in correspondence with the previous results by using other methods, such as the Kubo formula at the AdS boundary [39] and the membrane paradigm at the horizon [40]. Our ratio $\frac{\eta}{s_c}$ does not receive any correction from the third order Lovelock term which is the same result as shown in the Gauss-Bonnet gravity [11]. When $\tilde{\alpha}_2 \to 0$, the ratio $\frac{\eta}{s_c}$ reduces to $\frac{1}{4\pi}$ in Einstein gravity [5].

The kinematic viscosity ν is defined by $\frac{\eta}{\omega}$ and is expressed as

$$\nu = \frac{\Xi}{4\pi T_c}\left[1 - 2\tilde{\alpha}_2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right], \tag{52}$$

where

$$\Xi = \left[(n-1)r_h - \frac{(n-3)q^2}{r_h^{2n-5}}\right]\left[(n-1)r_h + \frac{(n-1)q^2}{r_h^{2n-5}} - \frac{2(n-2)q^2}{r_c^{n-3}r_h^{n-2}}\right]^{-1}. \tag{53}$$

The kinematic viscosity ν remains nonnegative when $\tilde{\alpha}_2 \leq \left[2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]^{-1}$. When $\tilde{\alpha}_2 = \left[2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]^{-1}$, we have the extremal case with $\nu = 0$ and the incompressible charged Navier-Stokes equation Eq. (7) becomes the nonrelativistic incompressible Euler's equation

$$\tilde{\partial}_\tau \beta_j + \beta^b \tilde{\partial}_b \beta_j + \tilde{\partial}_j P_r = \tilde{f}_j, \quad \tilde{\partial}_i \beta^i = 0. \tag{54}$$

Inserting the expansion of $\sqrt{f(r_c)}$, the kinematic viscosity ν can get influences from both Gauss-Bonnet and the third order Lovelock corrections,

$$\nu = \left[\Psi^{1/2}(r_c) + \tilde{\alpha}_2\Psi^{1/2}(r_c)\left(\frac{\Psi(r_c)}{2r_c^2} - 2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right) - \frac{\tilde{\alpha}_3}{6r_c^4}\Psi^{5/2}(r_c)\right]$$

$$\times \left[(n-1)r_h + \frac{(n-1)q^2}{r_h^{2n-5}} - \frac{2(n-2)q^2}{r_c^{n-3}r_h^{n-2}}\right]^{-1}. \tag{55}$$

Now let us look at the dimensionless coordinate invariant diffusivity $\bar{D}_c$. It was defined as the kinematic viscosity timing the temperature on the surface Σ_c [7] and it relates to the measured decay rate of the perturbation. The diffusivity reads

$$\bar{D}_c = T_c\nu = \frac{\Xi}{4\pi}\left[1 - 2\tilde{\alpha}_2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]. \tag{56}$$

Obviously, the diffusivity $\bar{D}_c$ is independent of the surface Σ_c. Only the Gauss-Bonnet correction appears if compared with the Einstein gravity case, while the third order Lovelock term does not show up in the diffusivity.

Considering the characteristic scale of the perturbation $L \sim \epsilon^{-1}$ and the velocity $\beta = \sqrt{\beta_i\beta^i} \sim \epsilon$, the Reynolds number of the dual fluid is given as

$$\mathcal{R}_e(r_c) = \frac{\beta L}{\nu} \propto \frac{1}{\nu}$$

$$= \frac{4\pi T_c}{\Xi}\left[1 - 2\tilde{\alpha}_2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right]^{-1}. \tag{57}$$

So the Reynolds number $\mathcal{R}_e(r_c)$ of the dual fluid is proportional to the local temperature T_c on the surface Σ_c. When the cutoff surface approaches the event horizon of the black brane background, the local temperature T_c as well as Reynolds number $\mathcal{R}_e(r_c)$ become larger and larger, and then the dual fluid will become unstable. Using the expansion for temperature T_c, the Reynolds number becomes

$$\mathcal{R}_e(r_c) = \left[\Psi^{-1/2}(r_c) - \tilde{\alpha}_2\Psi^{-1/2}(r_c)\left(\frac{\Psi(r_c)}{2r_c^2} + 2\left(\frac{n-1}{n-3} - \frac{q^2}{r_h^{2n-4}}\right)\right) + \frac{\tilde{\alpha}_3}{6r_c^4}\Psi^{3/2}(r_c)\right]$$

$$\times \left[(n-1)r_h + \frac{(n-1)q^2}{r_h^{2n-5}} - \frac{2(n-2)q^2}{r_c^{n-3}r_h^{n-2}}\right]. \tag{58}$$

We see that both the Gauss-Bonnet and the third order Lovelock correction terms appear in the Reynolds number.

According to the thermodynamic relation, the total entropy is

$$S = s_c V_{n-2}(r_c) = \frac{r_c^{n-2}}{T_c}(\omega - q_c\mu_c). \tag{59}$$

Following the procedures in [7, 12], we introduce an arbitrary null vector $\tilde{\zeta}^\mu$ with $\tilde{\zeta}^\mu \tilde{\partial}_\mu = \tilde{\partial}_\tau - \tilde{\partial}_{x^1}$ which is tangent to the surface Σ_c, and then we can verify

$$\partial_{r_c}\left(\frac{r_c^{n-2}}{T_c}\omega\right) = \frac{r_c^{n-2}}{8\pi G T_h}\tilde{\zeta}^\mu \tilde{\zeta}^\nu \left(G_{\mu\nu}^{(1)}(r_c) + \alpha_2 G_{\mu\nu}^{(2)}(r_c) + \alpha_3 G_{\mu\nu}^{(3)}(r_c)\right) \tag{60}$$

and

$$\partial_{r_c}\left(\frac{r_c^{n-2}}{T_c}q_c\mu_c\right) = \frac{r_c^{n-2}}{T_h}\tilde{\zeta}^\mu \tilde{\zeta}^\nu T_{\mu\nu}(r_c). \tag{61}$$

Finally, we have

$$\partial_{r_c}\left[\frac{r_c^{n-2}}{T_c}(\omega - q_c\mu_c)\right] = \frac{r_c^{n-2}}{16\pi G T_h}\tilde{\zeta}^\mu \tilde{\zeta}^\nu \left[2\left(G_{\mu\nu}^{(1)}(r_c) + \alpha_2 G_{\mu\nu}^{(2)}(r_c) + \alpha_3 G_{\mu\nu}^{(3)}(r_c)\right)\right.$$
$$\left. + 16\pi G T_{\mu\nu}(r_c)\right]. \tag{62}$$

Since in classical gravity there does not exist entropy outside the black brane, we have the radial isentropic flow. Considering the equation of motion (2), the equation $\partial_{r_c}S = 0 \iff \tilde{\zeta}^\mu \tilde{\zeta}^\nu W_{\mu\nu}(r_c) = 0$ can be obtained which implies the equivalence between the isentropy of the RG flow and a radial gravitational field equation of the third order Lovelock-Maxwell gravity.

V. SUMMARY

Based on the static black brane metric and using the two finite diffeomorphism transformations and nonrelativistic long-wavelength expansion, we have solved the bulk equations of motion at an arbitrary cutoff surface outside the horizon in the third order Lovelock gravity up to the second order of the expansion parameter. We have computed the stress-energy tensor of the dual fluid on the cutoff surface through the Brown-York tensor on the surface. We have shown that the dual fluid on Σ_c obeys an incompressible charged Navier-Stokes equation. The viscosity to entropy density ratio is cutoff surface independent and does not get modification from the third order Lovelock gravity influence. This agrees with the result obtained at the AdS boundary in the AdS/CFT correspondence [39]. In the near horizon limit, the result of η/s is also consistent with that obtained in the membrane paradigm [40]. The existence of RG flow may explain why the membrane paradigm 3 and the AdS infinity give the same result of η/s in the third order Lovelock gravity. Different from η/s, our result shows that the kinematic viscosity ν receives correction from the third order Lovelock term. It has been proved that the entropy flow equation along the radial coordinate is equivalent to the radial gravitational equations in Einstein [7] and Gauss-Bonnet gravity [12]. Here we have further proved that this property is also satisfied in the third order

Lovelock gravity. The influences of the thermodynamic parameters due to the higher curvature corrections we have observed in the third Lovelock gravity are interesting. They give us more understandings of relating the gravity theory to the dual fluid.

Acknowledgments

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